Journal of Applied Mechanics and Technical Physics, Vol. 48, No. 3, pp. 331–339, 2007

SPONTANEOUS SWIRLING IN AXISYMMETRIC MHD FLOWS OF AN IDEALLY CONDUCTING FLUID WITH CLOSED STREAMLINES

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UDC 532.516 + 538.4

The region of instability of the Hill–Shafranov viscous MHD vortex with respect to azimuthal axisymmetric perturbations of the velocity field is determined numerically as a function of the Reynolds number and magnetization in a linear formulation. An approximate formulation of the linear stability problem for MHD flows with circular streamlines is considered. The further evolution of the perturbations in the supercritical region is studied using a nonlinear analog model (a simplified initial system of equations that takes into account some important properties of the basic equations). For this model, the secondary flows resulting from the instability are determined.

Key words: *axisymmetric MHD vortex, stability, numerical calculation, spontaneous swirling.*

Introduction. We consider steady-state axisymmetric (poloidal) viscous flow in a bounded axisymmetric region that is sustained by the field of mass forces (axisymmetric, poloidal) and (or) by moving (in an appropriate manner) boundaries, on which the attachment condition is satisfied. In cylindrical coordinates,

$$\boldsymbol{r} = (z, r, \varphi), \qquad \boldsymbol{v}_0(z, r) = (w_0(z, r), u_0(z, r), 0).$$

Small axisymmetric perturbations (generally arbitrary) with nonzero azimuthal ($v_\varphi = v$) velocity component are introduced into this flow. The mass-force field and the boundary conditions are not perturbed. If the perturbations damp or their amplitude does not increase, the flow is steady-state and swirling does not occur. In the case of instability, the perturbations grow. If the evolution of the initial perturbations, by virtue of the exact nonlinear equations, results in flow (steady-state, periodic, unsteady random, turbulent) in which the average azimuthal velocity component is finite:

$$\langle v_\varphi \rangle = \int_0^{2\pi} v_\varphi(t, r, z, \varphi)\, d\varphi \neq 0,$$

and if the energy of rotational motion around the symmetry axis is comparable to the energy of the initial poloidal flow, we shall speak of the occurrence of spontaneous swirling.

The problem of spontaneous swirling was first formulated in [1] as follows: can rotationally symmetric flow occur in the absence of obvious external sources of rotation, i.e., under conditions where axisymmetric irrotational motion is possible *a priori*?

At present, there at least two points of view concerning the occurrence of spontaneous swirling. One of them is described above, and the second is consists of the following. Self-similar axisymmetric (conical) flows of the form $\boldsymbol{v} = (v_r, v_\theta, v_\varphi) = (f_r(\theta)/r, f_\theta(\theta)/r, f_\varphi(\theta)/r)$ or partially invariant Kármán-type flows $\boldsymbol{v} = (v_r, v_z, v_\varphi) = (rg_r(z), g_z(z), rg_\varphi(z))$ are considered. For the function $f(\theta)$ in the first case and function $g(z)$ in the second, ordinary differential equations are derived; for small Reynolds numbers, they have solutions only with poloidal components, and for Reynolds numbers exceeding a certain critical value, solutions with $v_\varphi \neq 0$ appear [2]. This mathematical fact is treated as the occurrence of spontaneous swirling or autorotation. However, it is obvious that in the flows

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considered, the velocity circulation $\Gamma = rv_\varphi$ does not vanish at infinity, and, therefore, there is a momentum flux that is an external source of rotation, which contradicts the initial formulation of the problem of spontaneous swirling.

A simple example of the occurrence of such autorotation is given below. Let us consider a radial flow of a fluid (which is at rest at infinity) through a porous wall of a circular cylinder r_0 (plane problem). Let the radial fluid velocity $v_r = -q(r/r_0)$ and let $v_\varphi = 0$ at $r = r_0$. Then, for the azimuthal-velocity circulation $\Gamma = rv_\varphi$ in the steady-state flow, we have the equation

$$\Gamma_{rr} = (1 - \mathrm{Re})\Gamma_r/r,$$

where $\mathrm{Re} = qr_0/\nu$ is the Reynolds number and ν is the viscosity. The general solution of this equation is $\Gamma = Ar^{2-\mathrm{Re}} + B$ ($\mathrm{Re} \neq 1$). From this it follows that for $\mathrm{Re} < 2$, flow bounded at infinity is possible only for $A = B = 0$, i.e., in the absence of rotation. At the same time, for $\mathrm{Re} > 2$, we have

$$v_\varphi = \Gamma_0(1 - (r_0/r)^{\mathrm{Re}-1})/r$$

(Γ_0 is a constant), which corresponds to rotational flow; furthermore, the fluid at infinity is at rest, and at $r = r_0$, the attachment condition is satisfied. However, if one requires that the velocity circulation at infinity vanish (i.e., if the angular momentum influx is eliminated), the rotation will be absent. This expanded treatment of the concept of spontaneous swirling seems invalid. Below, this term will only be used in the original sense.

A simple example of flow swirling is the occurrence of a bath tube vortex [3]. In this case, the mechanism generating rotational motion, just as in the case of intense mesoscale atmospheric vortices (dust piles, waterspouts, and tornado) is not completely understood. It is possible that spontaneous swirling plays an important role in this mechanism. The problem of spontaneous swirling is discussed in [1, 4], where examples are given of approximate solutions describing this given phenomenon. However, in the papers cited, a rotating fluid flows into the region considered, which leads one to doubt that obvious sources of rotation are absent.

A more rigorous formulation of the problem in question is given in [5, 6]. The formulation proposed in those papers provides a strict control of the kinematic flow of the axial component of the angular momentum to eliminate supply of the rotating fluid into the flow region. In [7], it is shown that in the case of MHD flows, the flux of the axial component of the angular momentum transferred by the magnetic field needs to be controlled and a formulation is proposed that eliminates the inflow of the indicated momentum components. In a viscous fluid, the occurrence of rotational flow is considered as a bifurcation of the initial rotationally symmetric flow to steady-state swirling flow as a result of instability [1], i.e., in this case, the steady-state problem for fixed boundary conditions has at least two solutions: without rotation and with swirling. Generally, secondary swirling flows that arise can also be unsteady (periodic and irregular self-oscillations, chaotic and turbulent flows).

For inviscid flows, this formulation is meaningless. In this case, in a bounded region there is a set of axisymmetric flows (without swirling) and rotationally symmetric flows (with swirling). Therefore, the occurrence of spontaneous swirling is considered as instability of the initial axisymmetric flow that leads to an increase in the azimuthal velocity amplitude and the kinetic energy of rotational motion due to the energy of the poloidal components (in the exact nonlinear formulation, their sum remains constant by virtue of the energy conservation law). The result is flow with finite energy of rotational motion.

We note that the occurrence of swirling does not violate the angular momentum conservation law. In an inviscid fluid, differential rotation with unchanged angular momentum occurs, and in a viscous fluid with the attachment condition on the boundaries of the flow region, the angular momentum is not necessarily conserved and rotational flow such as the Stokes vortex may occur.

The difficulties encountered in studies of three-dimensional flows have motivated the search for the simplest situations in which the phenomenon in question is possible. In this connection, stability against swirling have been studied for some steady-state rotationally symmetric flows subjected to rotationally symmetric perturbations.

It has been shown [5, 6] that the bifurcation of axisymmetric flow — the occurrence of rotationally symmetric flow — is absent in the case of an arbitrary compressible fluid of variable viscosity. For axisymmetric flows of a viscous incompressible fluid with finite conductivity in a magnetic field, it has been shown [7] that rotationally symmetrical spontaneous swirling is impossible if the cross section of the flow region by the meridional plane is simply connected. In such regions, the poloidal components of the magnetic field vanish with time because of the finite conductivity.

For an ideally conducting fluid, the nature of the connectedness of the flow region is of no significance since in rotationally symmetric flows of this fluid, the poloidal components of the magnetic field do not vanish because of freezing-in and, as is shown in [8, 9], under certain conditions there may be instability with respect to the initial swirling and an exponential or (in some cases) a linear growth of azimuthal perturbations with time.

The problems of instability in the viscous problem in a linear approximation, the nonlinearity effect, and the characteristics (structure) of the resulting secondary flow (viscous and inviscid) remain to be solved. The present paper is an attempt to solve these problems by studying the linear stability of axisymmetric viscous flows such as the Hill–Shafranov vortex and analog models of axisymmetric MHD flows with closed streamlines.

1. Linear Viscous Instability of Axisymmetric MHD Flow. In the conventional notation, flows of an ideally conducting viscous incompressible fluid in a magnetic field are described by the following system of equations (the fluid density is $\rho = 1$):

$$\boldsymbol{v}_t - \boldsymbol{v} \times \operatorname{rot} \boldsymbol{v} + \boldsymbol{h} \times \operatorname{rot} \boldsymbol{h} = \boldsymbol{F} + \nu \Delta \boldsymbol{v} - \nabla (p + \boldsymbol{v}^2/2); \tag{1.1}$$

$$\operatorname{div} \boldsymbol{v} = 0; \tag{1.2}$$

$$\boldsymbol{h}_t = \operatorname{rot} (\boldsymbol{v} \times \boldsymbol{h}); \tag{1.3}$$

$$\operatorname{div} \boldsymbol{h} = 0. \tag{1.4}$$

Here $\boldsymbol{h} = \boldsymbol{H}/\sqrt{4\pi}$, where $\boldsymbol{F} = (F_z, F_r, 0)$ is the poloidal field of the external-mass forces.

In the cylindrical coordinate system $\boldsymbol{v} = (u, v, w)$, the poloidal velocity components $\boldsymbol{h} = (h_1, h, h_3)$, $\boldsymbol{r} = (r, \varphi, z)$ of steady-state rotationally symmetric fluid flows of the type considered can generally be described with the use of Eqs. (1.2) and (1.4) by the relations

$$u = -\frac{\gamma(\psi)}{r} \frac{\partial \psi}{\partial z}, \qquad w = \frac{\gamma(\psi)}{r} \frac{\partial \psi}{\partial r}, \qquad h_1 = -\frac{\varepsilon(\psi)}{r} \frac{\partial \psi}{\partial z}, \qquad h_3 = \frac{\varepsilon(\psi)}{r} \frac{\partial \psi}{\partial r},$$

where $\gamma(\psi)$, $\varepsilon(\psi)$ are arbitrary dependences on the stream function ψ. Since for the flows considered below, the functions $\gamma(\psi)$ and $\varepsilon(\psi)$ are constant, one of them can be set equal to unity without loss of generality. Let $\gamma = 1$. Then, ε acquires the meaning of the coefficient of proportionality between the magnetic field and the poloidal velocity component $\boldsymbol{h}_p = \varepsilon \boldsymbol{v}_p$ in the initial flow. The quantity ε will be called the magnetization.

We consider a number of problems of the stability of steady-state rotationally symmetric flow ($v = 0$ and $h = 0$) with respect to swirling (the occurrence of rotationally symmetrical flow $v \neq 0$). In the linear approximation, the evolution of the azimuthal components of the velocity v and the magnetic field h is not related to the evolution of perturbations of the poloidal components and can be studied separately. From (1.1) and (1.3), we find that in the cylindrical coordinate system, v and h satisfy the equations

$$\begin{gathered} v_t + u(v - \varepsilon h)_r + w(v - \varepsilon h)_z + \frac{u}{r}(v - \varepsilon h) = \nu \Big(\frac{\partial^2 v}{\partial z^2} + \frac{\partial}{\partial r} \Big(\frac{1}{r} \frac{\partial (rv)}{\partial r} \Big) \Big), \\ h_t + u(h - \varepsilon v)_r + w(h - \varepsilon v)_z - (u/r)(h - \varepsilon v) = 0. \end{gathered} \tag{1.5}$$

Because flows in a bounded region are considered, the internal flow need not be matched to the external flow. This provides more possibilities for obtaining a broad class of exact analytical solutions that describe the initial flows. Moreover, the source of the main (initial) poloidal flow in the problem of spontaneous swirling is not as important; therefore, in describing the initial axisymmetric flow, as the stream function one can use an arbitrary, fairly regular function $\psi(r, z)$ that takes constant values on the closed lines in the meridional plane (r, z). Any such line can be used as the boundary of a toroidal region. In the presence of the corresponding mass forces and the motion of the boundaries, any flow defined in this manner can be treated as an exact steady-state axisymmetric solution of the Navier–Stokes MHD equations.

Investigation of the stability of these flows using both analytical and numerical methods involves considerable difficulties. Therefore, to elucidate the main features of the problem of swirling in flows with closed streamlines, along with the problem for the Hill–Shafranov vortex in a point (linear) formulation, we study stability for flows that are exact steady-state solutions in the above-mentioned sense in an approximate formulation (a narrow gap in flow with circular streamlines).

2. Hill–Shafranov Vortex. Let us consider the Hill–Shafranov vortex as the initial flow. For the poloidal velocity components in spherical coordinates, we have

$$\Psi = (1/2)r^2(1-r^2)\sin^2\theta, \qquad V_{0r} = (1-r^2)\cos\theta, \qquad V_{0\theta} = (2r^2-1)\sin\theta. \tag{2.1}$$

The steady-state flow (2.1) is an exact solution of the Navier–Stokes equations and is sustained by the corresponding motion of the spherical boundary. For the azimuthal components of the velocity and magnetic field, system (1.5) becomes

$$\begin{gathered}
\frac{\partial v}{\partial t} + (1-r^2)\cos\theta(v-\varepsilon h)_r + \frac{(2r^2-1)\sin\theta}{r}(v-\varepsilon h)_\theta + r\cos\theta(v-\varepsilon h) = \frac{1}{\mathrm{Re}} D^2 v,\\
\frac{\partial h}{\partial t} + (1-r^2)\cos\theta(h-\varepsilon v)_r + \frac{(2r^2-1)\sin\theta}{r}(h-\varepsilon v)_\theta - r\cos\theta(h-\varepsilon v) = 0,\\
D^2 v = \frac{1}{r}\frac{\partial}{\partial r}\Big(\frac{\partial\,(rv)}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\Big(\frac{1}{\sin\theta}\,\frac{\partial\,(v\sin\theta)}{\partial\theta}\Big).
\end{gathered} \tag{2.2}$$

We write the required functions as expansions in which the radial dependence is chosen so as to satisfy the boundary conditions $v(0,\theta) = v(1,\theta) = 0$ and $h(0,\theta) = 0$. On the z axis, the regularity condition (vanishing of the azimuthal components of the velocity and magnetic field) should be satisfied:

$$\begin{gathered}
v(t,r,\theta) = \sum_n v_n(t,r)P^{(1)}_{2n-1}(\cos\theta) = \sum_n\sum_m v_{m,n}\sin(2\pi mr)P^{(1)}_{2n-1}(\cos\theta)\,\mathrm{e}^{i\lambda t},\\
h(t,r,\theta) = \sum_n h_n(t,r)P^{(1)}_{2n-1}(\cos\theta) = \sum_n\sum_m h_{m,n} r\cos(2\pi mr)P^{(1)}_{2n-1}(\cos\theta)\,\mathrm{e}^{i\lambda t}.
\end{gathered} \tag{2.3}$$

Here $P^{(1)}_{2n-1}(\cos\theta)$ are first- order associated Legendre polynomials of odd degree. The Legendre polynomials of odd degree are chosen as trial functions because they provide regularity on the symmetry axis and also because the action of the operator D^2 on these functions gives a simple result by virtue of the corresponding recursive relations:

$$D^2 f(r)P^{(1)}_l(\cos\theta) = \Big(\frac{d^2}{dr^2} + \frac{2}{r}\,\frac{d}{dr} - \frac{l(l+1)}{r^2}\Big) f(r)P^{(1)}_l(\cos\theta).$$

To determine the critical values of the parameter ε and the Reynolds number Re above which the flow becomes unstable with respect to swirling, we formulate the problem for the eigenvalues $\lambda = \lambda(\varepsilon, \mathrm{Re})$. Substituting expansions (2.3) into Eqs. (2.2) and using the Galerkin procedure, we obtain an infinite system of linear algebraic equations for the unknown amplitudes. To obtain an approximate numerical solution, we discard all amplitudes v_{mn} and h_{mn} whose indices satisfy the inequality $2(n+m) > 2N_* + 3$. As a result, we obtain the system of algebraic equations

$$A\boldsymbol{x} = \lambda\boldsymbol{x},$$

where $\boldsymbol{x}$ is a one-dimensional complex vector composed of the required amplitudes and A is a complex matrix. The eigenvalues, i.e., the solution of the system of equations $\mathrm{Real}\,(\det(A-\lambda I)) = 0$, $\mathrm{Imag}\,(\det(A-\lambda I)) = 0$, were calculated using the modified Powell method with a finite-difference Jacobian [10]. As a result, we obtain a functional dependence of the eigenvalue on the Reynolds number. The critical values are determined by sequentially fixing the magnetization value ε and varying the Reynolds number Re until the minimum imaginary part in the spectrum obtained becomes negative, which indicates instability of the initial flow and an exponential growth of azimuthal perturbations. The calculations were performed for $N_* = 20$. Figure 1 shows the calculation results in the form of a curve that separates the stability and instability regions. The instability region is dashed. The occurrence of instability for $\varepsilon > 1$ is rather unexpected because for an inviscid fluid, instability occurs only for $0 < \varepsilon < 1$ [8]. The physical mechanism of this phenomenon (extension of the instability region due to the effect of viscosity) is unclear.

To draw a solid conclusion on the possibility of spontaneous swirling, it is necessary to consider initial perturbations that satisfy the initial condition $h = 0$ [7]. In the absence of proof of the completeness of the eigenfunctions, the spectral method does not allow this conclusion to be made. Under the above-mentioned additional condition, the instability region can change significantly (or even vanish); therefore, numerical calculations of the

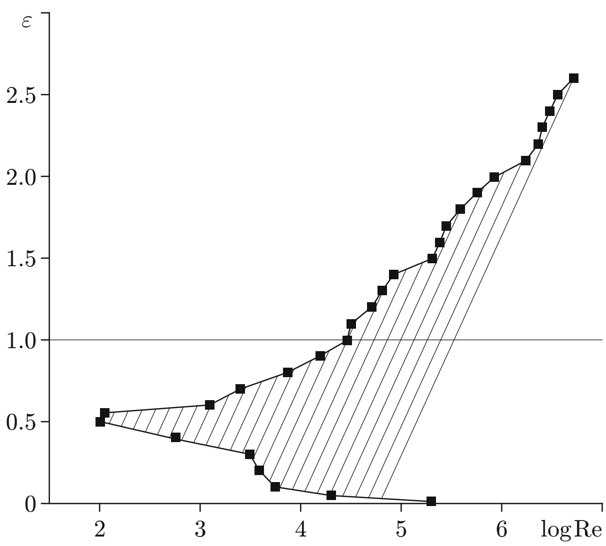

Fig. 1. Instability boundary.

problem with initial conditions in the supercritical region are required to verify the existence of instability when the critical Reynolds number exceeds the critical value. This is also valid for instability in the region $\varepsilon > 1$. In this region, the critical values of the Reynolds number are high, and numerical experiments involve serious computational difficulties.

3. Linear Stability of MHD Flow with Circular Streamlines. Let the stream function have the form $\psi(z,r) = \psi(R)$, where $R = \sqrt{(r-r_0)^2 + z^2}$ is the distance from the circular axis of the torus ($R < r_0$) and r_0 is the distance from the z axis to the common circular axis of toruses of small radius R. This flow can be treated as an exact steady-state solution of the Navier–Stokes MHD equations in the above-mentioned sense.

We introduce the azimuthal and radial (poloidal) velocity component. In the initial flow, we have

$$q(R,\theta) = w\cos\theta + u\sin\theta = -\frac{1}{r_0 - R\cos\theta}\,\frac{\partial\psi(R)}{\partial R},$$

$$p(R,\theta) = w\sin\theta - u\cos\theta = 0;$$

in the magnetic field,

$$g(r,\theta) = h_3\cos\theta + h_1\sin\theta = -\frac{\varepsilon}{r_0 - R\cos\theta}\,\frac{\partial\psi(R)}{\partial R},$$

$$f(R,\theta) = h_3\sin\theta - h_1\cos\theta = 0.$$

In the new variables (R,θ), system (1.5) is written as

$$v_t + \frac{q}{R}\,v_\theta + s(R,\theta)qv = \frac{g}{R}\,h_\theta + s(R,\theta)gv + \nu\Big(v_{RR} + \frac{1}{R}\,v_R$$

$$+ \frac{1}{R^2}\,v_{\theta\theta} - \frac{\cos\theta}{r_0 - R\cos\theta}\,v_R + \frac{\sin\theta}{R(r_0 - R\cos\theta)}\,v_\theta - \frac{v}{(r_0 - R\cos\theta)^2}\Big); \qquad (3.1)$$

$$h_t + \frac{q}{R}\,h_\theta - s(R,\theta)qh = \frac{g}{R}\,v_\theta - s(R,\theta)gv; \qquad (3.2)$$

$$s(r,\theta) = \frac{\sin\theta}{1 - r\cos\theta}, \qquad c(r,\theta) = \frac{\cos\theta}{1 - r\cos\theta}.$$

The flow considered is enclosed between the two streamlines corresponding to the values $R = R_0$ and $R = R_1$. The solution of system (3.1), (3.2) is subject to the boundary conditions: the periodicity of the functions v and h on the streamlines:

$$v(R,0) = v(R,2\pi), \qquad h(R,0) = h(R,2\pi)$$

and the attachment condition on the boundaries of the region considered:

$$v(R_0,\theta) = v(R_1,\theta) = 0.$$

Conditions are not imposed on the azimuthal component of the magnetic field h on the boundaries of the flow region, and its value is determined from Eq. (3.2).

4. Approximation of a Narrow Gap between the Boundary Streamlines. We consider the case where the distance between the boundary streamlines is small, i.e., $R_1 - R_0 \ll R_0$ ($R_1 > R_0$). We set $R = R_0(1+ay/\pi)$, where $a = (R_1 - R_0)/R_0 \ll 1$, so that the variable y varies in the interval $[0,\pi]$. Let $\psi(R) = -q_0 r_0 R$. Taking into account the smallness of the distance between the streamlines on the left side of Eqs. (3.1) and (3.2), we set $R = R_0$ and retain only the derivative with respect to R on the right of (3.1).

After the simplifications, we seek a solution of Eqs. (3.1) and (3.2) in the form $v = v(\theta)\sin(my)$, $h = h(\theta)\sin(my)$. This form of the solution automatically satisfy the boundary conditions on the streamlines and allows the system to be reduced to equations in which the required functions depend only on the variables θ and t. As a result, Eqs. (3.1) and (3.2) can be written in dimensionless form

$$v_t + \frac{1}{1-k\cos\theta}(v-\varepsilon h)_\theta + \frac{k\sin\theta}{(1-k\cos\theta)^2}(v-\varepsilon h) = -\frac{1}{\mathrm{Re}}\,v,$$

$$h_t + \frac{1}{1-k\cos\theta}(h-\varepsilon v)_\theta - \frac{k\sin\theta}{(1-k\cos\theta)^2}(h-\varepsilon v) = 0. \tag{4.1}$$

Here R_0/q_0 is the dimensionless unit of time, $k = R_0/r_0$, and $\mathrm{Re} = a^2 q_0 R_0/(\nu\pi^2 m^2)$.

The initial perturbation of the velocity introduced into the initial steady-state flow has the form

$$v(0,\theta) = v_0 + v_1\sin\theta, \qquad h(0,\theta) = 0,$$

where v_0 and v_1 are some constants.

The solutions of system (4.1) were found numerically from the initial data and the periodicity condition: $v(0) = v(2\pi)$, $h(0) = h(2\pi)$. The critical curves on the plane $(\varepsilon, \mathrm{Re})$ for various values of k that separate the regions of stability and instability with respect to swirling are similar to the curve shown in Fig. 1. As in the case of the Hill–Shafranov vortex, instability occurs in the region $\varepsilon > 1$.

To determine the subsequent development of the perturbations and the occurrence of spontaneous swirling, it is necessary to study the complete system of equations with allowance for nonlinearity in the instability region on the plane $(\varepsilon, \mathrm{Re})$. In the exact formulation, this problem is very complicated and its solution requires a powerful computational equipment.

The analog model studied in the present paper is described below. Numerical calculations of the problem subject to initial conditions show that oscillating (for an inviscid fluid) and steady-state (for a viscous fluid) secondary swirling flows occur in the instability region. In the case $\varepsilon = 1$ for an inviscid fluid, an exact solution is obtained. In the case of small but finite amplitudes for the steady-state secondary viscous flow at the stability limit, an approximate analytical solution is obtained.

5. Analog Model. We consider the following analog model of spontaneous swirling:

$$q_t + h^2 - v^2 = f(x,y) + \nu q_{yy},$$

$$v_t + v_x + qv = \varepsilon(h_x + qh) + \nu v_{yy}, \tag{5.1}$$

$$h_t + h_x - qh = \varepsilon(v_x - qv).$$

This system has a number of properties similar to the properties of the basic (exact) system. The function $q(x,y,t)$ in (5.1) models the periodic poloidal flow in the gap $0 < y < \pi$; $f(x,y)$ are the mass forces that sustain the initial poloidal flow. A solution of the system that is periodic in x is sought in the region $0 < x < 2\pi$, $0 < y < \pi$. On the lower and upper boundaries, conditions are imposed in accordance with the problem considered. For system (5.1) with viscosity $\nu = 0$, the energy conservation law (an analog of the energy conservation law in the exact formulation) holds.

Let us consider the inviscid nonlinear problem:

$$q_t + h^2 - v^2 = 0,$$

$$v_t + v_x + qv = \varepsilon(h_x + qh),$$

$$h_t + h_x - qh = \varepsilon(v_x - qv).$$

This system has a steady-state solution $q = q_0(x)$, where $q_0(x)$ is an arbitrary periodic function: $q_0(x+2\pi) = q_0(x)$. Let $\varepsilon = 1$ and let at $t = 0$, $q(0,x) = q_0(x)$, $v(0,x) = v_0(x) = v_0(x+2\pi)$, and $h(0,x) = 0$. We set $A = v + h$ and $B = v - h$. Then, for $\varepsilon = 1$, we obtain the system of equations

$$q_t = AB, \qquad A_t = -2qB, \qquad B_t = -2B_x$$

with the initial conditions $q(0,x) = q_0(x)$, $A(0,x) = v_0(x)$, and $B(0,x) = v_0(x)$.

If $v_0(x) = V_0 = \text{const}$, we have

$$B = V_0, \qquad q_t = V_0 A, \qquad A_t = -2V_0 q.$$

In view of the initial conditions, this implies

$$A(t,x) = -\sqrt{2}\, q_0(x) \sin(\sqrt{2}\, V_0 t) + V_0 \cos(\sqrt{2}\, V_0 t), \qquad B(t,x) = V_0,$$

$$q(t,x) = q_0(x) \cos(\sqrt{2}\, V_0 t) + (V_0/\sqrt{2}\,) \sin(\sqrt{2}\, V_0 t).$$

For v and h, we obtain

$$v(x,t) = -q_0(x) \sin(\sqrt{2}\, V_0 t)/\sqrt{2} + V_0(1 + \cos(\sqrt{2}\, V_0 t))/2,$$

$$h(x,t) = -q_0(x) \sin(\sqrt{2}\, V_0 t)\sqrt{2} - V_0(1 - \cos(\sqrt{2}\, V_0 t))/2.$$

In a linear approximation for the same initial conditions, the functions v and h increase in proportion to time t as a result of instability. In this example, nonlinearity results in oscillating secondary flow, whose period is the larger the smaller the initial perturbation amplitude, and the oscillation frequency is determined by the intensity of the initial flow. A numerical study of this problem for $0 < \varepsilon < 1$ shows that the instability results in the occurrence of an oscillating regime which is irregular in time and on the spatial coordinate and whose amplitude depends on the amplitude of the initial flow and is comparable to it in magnitude. The same behavior of the perturbations can be expected in the exact formulation.

Let us consider the analog model taking into account viscosity. A simple example of such flow can be constructed retaining only the dependence on the coordinate y ($0 \le y \le \pi$). Then, system (5.1) is simplified:

$$q_t + (h^2 - v^2) = \nu q_{yy}; \tag{5.2}$$

$$v_t + q(v - \varepsilon h) = \nu v_{yy}; \tag{5.3}$$

$$h_t - q(h - \varepsilon v) = 0 \tag{5.4}$$

($1/\nu$ is an analog of the Reynolds number). System (5.2)–(5.4) has the solution $q = q_0$, $v = h = 0$ (an analog of steady flow without swirling). Let us examine the stability of this solution in a linear approximation. We set $q = q_0 + q'$. Then, the linear system is written as

$$q'_t = \nu q'_{yy}; \tag{5.5}$$

$$v_t + q_0 v = \varepsilon q_0 h + \nu v_{yy}; \tag{5.6}$$

$$h_t - q_0 h = -\varepsilon q_0 v, \tag{5.7}$$

where q', v, and h are small compared to q_0. The flow is considered in the region $0 \le y \le \pi$ with the boundary conditions

$$y = 0: \qquad q'(0) = 0, \qquad v(0) = 0,$$

$$y = \pi: \qquad q'(\pi) = 0, \qquad v(\pi) = 0.$$

Boundary conditions are not imposed on the function h since these values are defined by Eq. (5.7).

The initial conditions are given by

$$t=0: \qquad q'(0,y)=f_0(y), \qquad v(0,y)=v_0(y), \qquad h(0,y)=0.$$

From Eq. (5.5) it follows that q' tends to zero for any function $f_0(y)$. In this case (because the equations are simple), the evolution of v and h can be determined by employing the method of separation of variables.

We seek a solution in the form of the sums of the terms

$$v_n(t,y)=V_n(t)\sin(ny), \qquad h_n(t,y)=H_n(t)\sin(ny).$$

For $V_n(t)$ and $H_n(t)$, we obtain the system

$$V_n'=-(q_0+\nu n^2)V_n+\varepsilon q_0 H_n, \qquad H_n'=q_4 H_n-\varepsilon q_0 V_n$$

with the initial conditions $V_n(0)=V_{n0}$ and $H_n(0)=0$. The general solution of this system has the form

$$V_n(t)=a\exp(\lambda_+ t)+b\exp(\lambda_- t),$$

$$H_n(t)=\frac{\lambda_+ +q_0+\nu n^2}{\varepsilon q_0}\,a\exp(\lambda_+ t)+\frac{\lambda_- +q_0+\nu n^2}{\varepsilon q_0}\,b\exp(\lambda_- t),$$

where

$$\lambda_{+,-}=-\nu n^2/1\pm\sqrt{(q_0+\nu n^2/2)^2-\varepsilon^2 q_0{}^2}.$$

From the initial conditions, we have

$$a=k_-V_{n0}/(k_- -k_+), \qquad b=-k_+V_{n0}/(k_- -k_+).$$

Here

$$k_+=(\lambda_+ +q_0+\nu n^2)/(\varepsilon q_0), \qquad k_-=(\lambda_- +q_0+\nu n^2)/(\varepsilon q_0).$$

Finally, we obtain

$$V_n(t)=\frac{V_{n0}}{k_- -k_+}\,[k_-\exp(\lambda_+ t)-k_+\exp(\lambda_- t)],$$

$$H_n(t)=\frac{k_+k_-}{k_- -k_+}\,V_{n0}[\exp(\lambda_+ t)-\exp(\lambda_- t)].$$

Thus, for $\varepsilon^2<1+\nu n^2/q_0$, exponential instability takes place. This inequality implies (at least, in a linear formulation for $q_0>0$) that the viscosity reduces stability (instability arises at a larger magnetic field). To some extent, this result explains the instability obtained above for the Hill–Shafranov MHD vortex in the case $\varepsilon>1$.

6. Nonlinear Steady-State Viscous Flow. Numerical investigation of the problem with the initial conditions for the nonlinear model (5.2)–(5.4) shows that for fairly large values of $\mu=1/\nu$ (but not too large) steady-state secondary flow with an amplitude comparable to the amplitude of the initial flow q_0 occurs. In the steady-state flow, $h=\varepsilon v$. According to this, to determine q and v, we obtain the system of ordinary differential equations

$$q''=-\mu(1-\varepsilon^2)v^2, \qquad v''=\mu(1-\varepsilon^2)qv \tag{6.1}$$

with the boundary conditions $q(0)=q(\pi)=-1$ and $v(0)=v(\pi)=0$. Here the primes denote differentiation with respect to y. This problem has a solution $q(y)=-1$, $v(y)=0$ (flow without swirling).

We set $q=-(1-u)$. Then, system (6.1) becomes

$$u''=-\mu(1-\varepsilon^2)v^2, \qquad v''=-\mu(1-\varepsilon^2)(1-u)v.$$

The boundary conditions are $u(0)=u(\pi)=0$ and $v(0)=v(\pi)=0$.

In a linear approximation for $(1-\varepsilon^2)\mu=1$, there exists a steady-state solution $v(y)=A\sin y$, $u=0$, where the amplitude A is arbitrary.

Assuming that the solution exists for $(1-\varepsilon^2)\mu>1$ [and $(1-\varepsilon^2)\mu-1\ll 1$], we seek a solution of the nonlinear problem in the following form ($\alpha\ll 1$):

$$v=\alpha v_1+\alpha^3 v_3+\ldots\ , \qquad u=\alpha^2 u_2+\ldots\ , \qquad \mu=1+\alpha^2\mu_2+\ldots\ .$$

Next, using the standard method (from the condition of resolvability of the equations for v_3 and u_4), we obtain

$$v(y) = \sqrt{(3\pi/8)[(1-\varepsilon^2)\mu - 1]}\,\sin y + \dots\ ,$$

$$u(y) = (3\pi/32)[(1-\varepsilon^2)\mu - 1][\sin y^2 - y(y-\pi) + \dots].$$

Thus, for the analog model, it is shown that at the stability boundary for Reynolds numbers exceeding the critical value, instability results in steady-state secondary swirling flow.

Conclusions. The studies performed revealed linear instability of viscous MHD flows such as the Hill–Shafranov vortex in a bounded region. The boundary of the instability region was determined as a function of the magnetization and the Reynolds number of the initial flow. The possibility of spontaneous swirling was shown for the problem considered in the exact formulation, although the results obtained did not provide the ultimate answer. However, the investigation of the analog model, for which the secondary flows resulting from instability were found, allows one to expect the occurrence of a secondary regime in the exact formulation and even to predict its characteristics to some extent.

REFERENCES

1. M. A. Gol'dshtik, E. M. Zhdanova, and V. N. Shtern, "Spontaneous swirling of a submerged jet," *Dokl. Akad. Nauk SSSR*, **277**, No. 4, 815–818 (1984).
2. M. A. Gol'dshtik, V. N. Shtern, and N. I. Yavorskii, *Viscous Flows with Paradoxical Properties* [in Russian], Nauka, Novosibirsk (1989).
3. M. A. Lavrent'ev and B. V. Shabat, *Problems of Hydrodynamics and Their Mathematical Models* [in Russian], Nauka, Moscow (1973).
4. A. M. Sagalakov and A. Yu. Yudintsev, "Three-dimensional self-oscillating magnetohydrodynamic flows of a fluid of finite conductivity in an annular channel in the presence of a longitudinal magnetic field," *Magn. Gidrodin.*, No. 1, 41–48 (1993).
5. B. A. Lugovtsov, "Is spontaneous swirling of axisymmetric flow possible?" *J. Appl. Mech. Tech. Phys.*, **35**, No. 2, 207–210 (1994).
6. Yu. G. Gubarev and B. A. Lugovtsov, "Spontaneous swirling in axisymmetric flows," *J. Appl. Mech. Tech. Phys.*, **36**, No. 4, 52–59 (1995).
7. B. A. Lugovtsov, "Spontaneous swirling in axisymmetric flows of a conducting fluid in a magnetic field," *J. Appl. Mech. Tech. Phys.*, **37**, No. 6, 802–809 (1996).
8. B. A. Lugovtsov, "Axisymmetric spontaneous swirling in an ideally conducting fluid in a magnetic field," *J. Appl. Mech. Tech. Phys.*, **38**, No. 6,839–841 (1997).
9. B. A. Lugovtsov, "Rotationally symmetrical spontaneous swirling in MHD flows," *J. Appl. Mech. Tech. Phys.*, **41**, No. 5, 870–878 (2000).
10. J. M. Ortega and W. C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press (1970)